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On resolving the multiplicity of arbitrary tensor products of the U(N) groups

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Abstract. Representations of U(N) are realised as right translations on holomorphic Hilbert (Bargmann) spaces of $n \times N$ complex variables. r-fold tensor product spaces of irreducible representations of U(N) are shown to be isomorphic to subspaces of the holomorphic Hilbert spaces. Maps are exhibited which carry an irreducible representation of U(N) into these subspaces. The algebra of operators commuting with these maps is constructed and it is shown how eigenvalues of certain of these operators can be used to resolve the multiplicity. Several examples from U(3) are explicitly worked out.

1. Introduction

One of the outstanding problems in the representation theory of compact groups is the multiplicity problem. In decomposing tensor products of representations of a group, the same irreducible representation may appear more than once; the problem is to find a canonical way of treating the equivalent representations that occur in this decomposition. Biedenharn and his co-workers have analysed this problem for the unitary groups from several different points of view (Moshinsky 1963, Brody *et al* 1965, Louck 1970 and references therein). One possibility is to make use of tensor operators; such an approach has recently been used in conjunction with holomorphic Hilbert (Bargmann) spaces of the type that will be used in this paper (Le Blanc and Rowe 1985, 1986, Le Blanc and Hecht 1987). Another possibility is to embed the tensor product space in a much larger space which provides a way of breaking the multiplicity (Biedenharn and Flath 1984). In these papers the tensor product space is usually a twofold tensor product.

In this paper we will give a general procedure for decomposing r-fold tensor products of irreducible representations of U(N) by exhibiting a general class of Casimir operators that commute with the U(N) action on the tensor product space. The eigenvalues of these operators can then be used to break the multiplicity. The main tools needed to carry out this analysis are a Fock space in $n \times N$ complex variables which is a carrier space for the tensor products, a Frobenius reciprocity type theorem proved by Klink and Ton-That (1988a) which can be used to bound and eventually to compute the multiplicity, and the theory of dual pairs (Moshinsky and Quesne 1970, Howe 1985), which is used to construct the Casimir operators. Our work follows in the spirit of Želobenko (1970) and in particular we generalise his notion of cycles for Casimir operators.

The irreducible representations of U(N) are labelled by N non-negative integers $(m)=(m_1,\ldots,m_N)$. In this paper we generalise the results of Klink and Ton-That (1988a), where the tensor products of only the simplest representations of the form $(m,0,\ldots,0)$ were considered; here we will deal with r-fold tensor products of arbitrary irreducible representations of U(N). Because we will rely heavily on the results of Klink and Ton-That (1988a) these results will be reviewed in § 2.

Section 3 makes use of the notion of dual pairs to exhibit an algebra of operators that commute with the U(N) action on the tensor product space. Theorem 3.1 gives the explicit form of this algebra of operators, while proposition 3.5 shows that these operators are Hermitian; the eigenvalues of elements of this algebra can then be used to resolve the multiplicity. This is shown explicitly in § 4 with examples of representations of U(3). In particular we show that the multiplicity occurring in the tensor product of the eight-dimensional representation with itself can be resolved with the eigenvalues of a Casimir operator, rather than with the symmetric and antisymmetric representations of the permutation group on two letters, as is usually done.

2. The general setup for tensor product decompositions

Let G denote the general linear group GL(N, $\mathbb{C}$). In general, a concrete realisation of a finite-dimensional irreducible representation of G can be obtained in the following fashion. If $(m)=(m_1,\ldots,m_N)$ is an N-tuple of integers which satisfy the *dominant* condition $m_1\geqslant m_2\geqslant\ldots\geqslant m_N\geqslant 0$, and if B denotes the subgroup of lower triangular matrices of G, then we define a holomorphic character $\pi^{(m)}: B\to\mathbb{C}^*$ by setting $\pi^{(m)}(b)=b_{11}^{m_1}\ldots b_{NN}^{m_N}$. Let $V^{(m)}$ denote the complex vector space of all polynomial functions $f:\mathbb{C}^{N\times N}\to\mathbb{C}$ which satisfy the covariant condition $f(bZ)=\pi^{(m)}(b)f(Z)$ for all (b,Z) belonging to $B\times\mathbb{C}^{N\times N}$. Let $R^{(m)}$ denote the holomorphically induced representation of G on $V^{(m)}$ defined by $[R^{(m)}(g)f](Z)=f(Zg)$, $g\in$ G; then according to the Borel-Weil theorem $R^{(m)}$ is irreducible and its highest weight is indexed by (m) which is called the *signature* of the representation $R^{(m)}$. Moreover, if we restrict this representation to the unitary group U(N), it remains irreducible. Finally, if we equip $V^{(m)}$ with the 'differentiation' inner product

$$(f,f')=f(D)\overline{f'(\bar{Z})}|_{Z=0} \tag{2.1}$$

where $f(D)$ denotes the differential operator obtained by replacing Z_{ij} by the partial derivative $\partial/\partial Z_{ij}$ $(1\leqslant i,j\leqslant N)$, then the representation of U(N) on $V^{(m)}$ is unitary (cf Klink and Ton-That 1988a).

Next we consider tensor products of r irreducible representations of G with signatures $(M_{(1)}),\ldots,(M_{(r)})$, respectively, where each label $(M_{(i)})$ is an N-tuple of integers $(M_{i_1},\ldots,M_{i_N})$, $1\leqslant i\leqslant r$. We discard those M_{ij}, $1\leqslant i\leqslant r$, $1\leqslant j\leqslant N$ which are equal to zero, and relabel the indices so that they form an n-tuple of integers of the form $(M_1,\ldots,M_{p_1},M_{p_1+1},\ldots,M_{p_2},M_{p_2+1},\ldots,M_{p_r})$, where $M_1,\ldots,M_{p_1}$ are the p_1 non-zero elements of $(M_{(1)})$, and $M_{p_1+1}\ldots M_{p_2}$ are the p_2 non-zero elements of $(M_{(2)})$, etc, such that the sum $p_1+\ldots+p_r=n$. Let $\mathbb{C}^{n\times N}$ denote the complex vector space of all $n\times N$ matrices. If $Z=(Z_{ij})$ is an element of $\mathbb{C}^{n\times N}$, we let $\bar{Z}$ denote its conjugate and write $Z_{ij}=X_{ij}+\sqrt{-1}\,Y_{ij}$; $1\leqslant i\leqslant n$, $1\leqslant j\leqslant N$. If $\mathrm{d}X_{ij}$ (respectively $\mathrm{d}Y_{ij}$) denotes the Lebesque measure on $\mathbb{R}$ we let $\mathrm{d}Z=\Pi\,\mathrm{d}X_{ij}\,\mathrm{d}Y_{ij}$ denote the Lebesque product measure $1\leqslant i\leqslant n$, $1\leqslant j\leqslant N$ on $\mathbb{R}^{nN}$. Define a Gaussian measure on $\mathbb{C}^{n\times N}$ by

$$\mathrm{d}\mu(Z)=\pi^{-nN}\exp[-\mathrm{Tr}(Z\bar{Z}')]\,\mathrm{d}Z$$

where Tr denotes the trace of a matrix. A mapping $f:\mathbb{C}^{n\times N}\to\mathbb{C}$ is a *holomorphic square-integrable* function if it is holomorphic on the entire domain $\mathbb{C}^{n\times N}$ and if

$$\int_{\mathbb{C}^{n\times N}} |f(Z)|^2\, \mathrm{d}\mu(Z)<\infty.$$

It is obvious that the holomorphic square-integrable functions form a complex vector space; in fact, they form a Hilbert space with the inner product

$$(f_1,f_2)=\int_{\mathbb{C}^{n\times N}} f_1(Z)\overline{f_2(Z)}\, \mathrm{d}\mu(Z). \tag{2.2}$$

It is easy to verify that this inner product actually coincides with the 'differentiation' inner product (2.1) if we replace polynomials by entire series expansions in Z_{ij}. Let $\mathscr{F}\equiv\mathscr{F}(\mathbb{C}^{n\times N})$ denote this Fock space.

If D_n denotes the group of all complex diagonal invertible matrices of order (n) and if $(M)=(M_1,\ldots,M_{p_1},\ldots,M_{p_r})$ is the n-tuple of positive integers we define a holomorphic character $\pi^{(M)}:D_n\to\mathbb{C}^*$ by

$$\pi^{(M)}(d)=d_{11}^{M_1}\ldots d_{nn}^{M_n}\qquad d=\begin{bmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn}\end{bmatrix}\in D_n.$$

A polynomial function $p:\mathbb{C}^{n\times N}\to\mathbb{C}$ is said to *transform covariantly with respect to* $\pi^{(M)}$ if

$$p(\mathrm{d}Z)=\pi^{(M)}(d)p(Z)\qquad \forall(d,Z)\in D_n\times\mathbb{C}^{n\times N}.$$

It is obvious that the polynomial functions which transform covariantly with respect to $\pi^{(M)}$ form a subspace of $\mathscr{F}$. We will denote this subspace by $\mathscr{P}^{(M)}$. Let $R^{(M)}$ denote the representation of G on $\mathscr{P}^{(M)}$ defined by $[R^{(M)}(g)p](Z)=p(Zg)$, $(\forall(Z,g)\in\mathbb{C}^{n\times N}\times$ G, $p\in\mathscr{P}^{(M)})$. In corollary 2.9 of Klink and Ton-That (1988a) we have proved the following Frobenius reciprocity type theorem.

Theorem 2.1. (a) If $n\leqslant N$ then the frequency of occurrence of the irreducible representation of G with signature $(m_1,m_2,\ldots,m_N)$ in $\mathscr{P}^{(M)}$ is equal to the dimension of the weight space $(M_1,\ldots,M_n,0,\ldots,0)$ in the representation $(m_1,\ldots,m_N)$ of G.

(b) If $n>N$ then the frequency of occurrence of the irreducible representation of G with signature $(m_1,\ldots,m_N)$ in $\mathscr{P}^{(M)}$ is equal to the dimension of the weight space $(M_1,\ldots,M_n)$ in the representation $V^{(m_1,\ldots,m_N,0,\ldots,0)}$ of G.

Using the Gelfand-Žetlin basis for $V^{(m_1,\ldots,m_N)}$ we see that theorem 2.1 allows us to compute the multiplicity of a representation with signature $(m_1,\ldots,m_N)$ in $\mathscr{P}^{(M)}$.

Set $\mathrm{G}'=\mathrm{GL}(n,\mathbb{C})$; then G′ acts on $\mathbb{C}^{n\times N}$ by left multiplication, and this action induces an action, denoted by L, of G′ in $\mathscr{F}$:

$$[L(g')f](Z)=f((g')^{-1}Z)\qquad \forall(Z,g')\in\mathbb{C}^{n\times N}\times G'\qquad \forall f\in\mathscr{F}.$$

We say that G and G′ are dual (cf Moshinsky and Quesne 1970, Howe 1985). In general, $\mathscr{P}^{(M)}$ is not invariant under the left action L of G′, but if we set $|M|=M_1+\ldots+M_n$ and define $\mathscr{P}^{|M|}$ to be the subspace of $\mathscr{F}$ which consists of all homogeneous polynomial of degree $|M|$, i.e. $f \in \mathscr{P}^{|M|}$ if and only if $f(\lambda Z) = \lambda^{|M|} f(Z)$ for all $\lambda \in \mathbb{C}$, then $\mathscr{P}^{|M|}$ is invariant under both actions R and L. We define the *isotypic component* of the G module $V^{(m_1,\ldots,m_N)}$ in $\mathscr{F}$ to be the sum of all G submodules in $\mathscr{F}$ which are equivalent to $V^{(m_1,\ldots,m_N)}$. In corollary 2.14 of Klink and Ton-That (1988a) we proved the following theorem.

Theorem 2.2. If $\mathscr{P}^{(M)}$ contains a submodule isomorphic to $V^{(m)}$ then the isotypic component $\mathscr{J}(V^{(m)})$ is contained in $\mathscr{P}^{|M|}$.

Now G′ contains the subgroup $\mathrm{K}' = \mathrm{GL}(p_1, \mathbb{C}) \times \ldots \times \mathrm{GL}(p_r, \mathbb{C})$ which consists of all elements of the form

$$\begin{pmatrix} k_1' & & 0 \\ & \ddots & \\ 0 & & k_r' \end{pmatrix} \qquad k_i' \in \mathrm{GL}(p_i, \mathbb{C}),\ 1 \leq i \leq r.$$

If B_{p_i} denotes the lower triangular (Borel) subgroup of $\mathrm{GL}(p_i, \mathbb{C})$ we let $H^{(M)}(\mathbb{C}^{n\times N}) \equiv H^{(M)}$ denote the subspace of $\mathscr{P}^{(M)}$ which consists of all polynomial functions f which satisfy the covariant condition

$$f\left(\begin{pmatrix} b_{p_1} & & 0 \\ & \ddots & \\ 0 & & b_{p_r} \end{pmatrix} Z\right) = b_{11}^{M_1} \ldots b_{nn}^{M_n} f(Z) \tag{2.3}$$

where b_{jj} $(1 \leq j \leq n)$ denotes the jth diagonal entry of the matrix

$$\begin{pmatrix} b_{p_1} & & 0 \\ & \ddots & \\ 0 & & b_{p_r} \end{pmatrix}.$$

By an argument similar to the one used in the proof of theorem 2.7 of Klink and Ton-That (1988a) we have the following theorem.

Theorem 2.3. If $V^{(M_{(i)})} = V^{(M_{i_1}, M_{i_2}, \ldots)}$ denote irreducible representations of G then the Kronecker tensor product $V^{(M_{(1)})} \otimes \ldots \otimes V^{(M_{(r)})}$ is isomorphic to the G module $H^{(M)}$.

From theorems 2.1, 2.2, and 2.3 we see that the theory of dual pairs and the G submodules $\mathscr{P}^{(M)}$ and $\mathscr{P}^{|M|}$ will play an important role in the decomposition of the tensor product $V^{(M_{(1)})} \otimes \ldots \otimes V^{(M_{(r)})}$. It should also be pointed out that theorem 2.1 gives an upper bound for the multiplicity of an irreducible representation of G occurring in the tensor product.

3. The explicit decomposition of the tensor product

We will now give a procedure for explicitly decomposing the tensor product (or equivalently the G module $H^{(M)}$) which includes the crucial step of multiplicity breaking. For this we let L_{ij} (respectively R_{rs}) denote the infinitesimal operators of L (respectively R) corresponding to the standard basis e_{ij} (respectively e_{rs}) of the Lie algebra $\mathbb{C}^{n\times n}$ (respectively $\mathbb{C}^{N\times N}$) of G′ (respectively G); then

$$L_{ij}=\sum_{l=1}^{N} Z_{il}\frac{\partial}{\partial Z_{jl}} \qquad R_{rs}=\sum_{k=1}^{n} Z_{kr}\frac{\partial}{\partial Z_{ks}} \qquad 1\leq i,j\leq n,\ 1\leq r,s\leq N. \tag{3.1}$$

Amongst these infinitesimal operators L_{ij} we have the particular operators $L_{i_pj_p}$, where $p=p_1,\ldots,p_r$, which correspond to the infinitesimal operators of the subgroups $\mathrm{GL}(p_i,\mathbb{C})$, $1\leq i\leq r$, of G′. We have now a very simple characterisation of the G module $H^{(M)}$. Indeed, it is easy to verify that $H^{(M)}$ consists of polynomial functions in $\mathcal{P}^{(M)}$ which are simultaneously annihilated by all lowering operators of the form

$$L_{i_pj_p} \qquad \text{with } i_p<j_p \text{ and all } p=p_1,p_2,\ldots,p_r. \tag{3.2}$$

Condition (3.2) is just the infinitesimal version of the Borel condition (2.3). Note that in (3.1) for convenience we let L_{ij} correspond to the standard basis e_{ji} so that the Borel condition corresponds to (3.2) with $i_p<j_p$ instead of the familiar condition $i_p>j_p$.

Now we assume that the number of times a representation $R^{(m)}$ of G occurs in $H^{(M)}$ is known (this is the Clebsch–Gordan series problem; there are a number of ways of obtaining a closed-form formula for this multiplicity (see Kostant 1959, Steinberg 1961, Koike 1988)). Actually, in our procedure of multiplicity breaking, we will show how to derive this multiplicity for each concrete example, which is a very interesting fact, since the closed-form formulae for multiplicity are in general impractical if not impossible to apply when the order of the group G is large. In contrast, to compute the multiplicity of $R^{(m)}$ in $\mathcal{P}^{(M)}$ using theorem 2.1 is a straightforward procedure which involves only the Gelfand–Žetlin tableaux. And, as we shall see, the multiplicity of $R^{(m)}$ in $\mathcal{P}^{(M)}$ together with projection operators $L_{i_pj_p}$ of (3.2) lead us immediately to the multiplicity of $R^{(m)}$ in $H^{(M)}$.

Now the infinitesimal operators L_{ij}, $1\leq i,j\leq n$, in (3.1) form a basis for the Lie algebra (of the group G′) with commutation relations

$$[L_{ij},L_{uv}]=\delta_{ju}L_{iv}-\delta_{iv}L_{uj} \qquad 1\leq i,j,u,v\leq n.$$

These basis elements generate a universal enveloping algebra $\mathcal{U}$ of right invariant differential operators which also acts on $\mathcal{F}$. Moreover, by the Poincaré–Birkoff–Witt theorem the ordered monomials in L_{ij} form a basis for the algebra $\mathcal{U}$.

Now suppose the G module $(R^{(m)}, V^{(m)})$ occurs in $\mathcal{P}^{(M)}$ with multiplicity μ. Then from the theory of reductive dual pairs (Moshinsky and Quesne 1970, Howe 1985) and from a consequence of Burnside's theorem (cf Dixmier 1974) there exist μ linearly independent elements in $\mathcal{U}$ which form a basis for the vector space $\mathrm{Hom}_G(V^{(m)},\mathcal{P}^{(M)})$ of all intertwining operators from $V^{(m)}$ to $\mathcal{P}^{(M)}$. In a forthcoming paper we will give a systematic procedure for obtaining these μ intertwining maps in concrete examples. In fact, since the algebra $\mathcal{U}$ consists of right-invariant differential operators to simplify our exposition it suffices to consider the highest-weight vector $h^{(m)}_{\max}$ of $V^{(m)}$; then it follows that we can choose μ elements $p_1(L_{ij}),\ldots,p_\mu(L_{ij})$ of $\mathcal{U}$ such that $p_l(L_{ij})h^{(m)}_{\max}$, $1\leq l\leq\mu$, are linearly independent highest-weight vectors of the μ copies of the G module isomorphic to $V^{(m)}$ which are contained in $\mathcal{P}^{(M)}$. Now to obtain an orthogonal direct sum decomposition of $\mathcal{J}(V^{(m)})\cap H^{(M)}$, the intersection of the isotypic component of $V^{(m)}$ with $H^{(M)}$, we must find operators in $\mathcal{U}$ that commute with the action of the subgroup $\mathrm{K}'=\mathrm{GL}(p_1,\mathbb{C})\times\ldots\times\mathrm{GL}(p_r,\mathbb{C})$ (or, equivalently, that commute with the operators $L_{i_pj_p}$ in (3.2) but without the condition $i_p<j_p$), and that decompose

'canonically' $\mathscr{J}(V^{(m)}) \cap H^{(M)}$ into distinct eigenspaces. For this, let $W_{\max}^{(m)(M)}$ denote the vector space spanned by $p_l(L_{ij})h_{\max}^{(m)}$ and let $\ker_{\max}^{(m),(M)}$ denote its projection in $H^{(M)}$, i.e. $\ker_{\max}^{(m)(M)}$ is the common kernel subspace in $W_{\max}^{(m)(M)}$ of all the operators $L_{i_p j_p}$ in (3.2). We will find operators in $\mathscr{U}$ which commute with the operators $L_{i_p j_p}$ and which decompose $\ker_{\max}^{(m)(M)}$ into distinct one-dimensional eigenspaces. For this, let us concentrate on the action of K′ and its right dual action on $\mathscr{F}$. Let Z be an element of $\mathbb{C}^{n\times N}$ and write Z in block form as

$$Z_1 = \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix}$$

where Z_i is a $p_i \times N$ matrix, $1 \leqslant i \leqslant r$. The action of K′ on Z is simply of the form

$$(k'_1, \ldots, k'_r) \to \begin{bmatrix} k'_1 Z_1 \\ \vdots \\ k'_2 Z_r \end{bmatrix} \qquad \text{for } (k'_1, \ldots, k'_r) \in \mathrm{K}'.$$

Its dual action is therefore

$$(g_1, \ldots, g_r) \to \begin{bmatrix} Z_1 g_1 \\ \vdots \\ Z_r g_r \end{bmatrix} \qquad \text{for all } (g_1, \ldots, g_r) \in \underbrace{G \times \ldots \times G}_{r}.$$

According to the theory of dual pairs our task is to find operators in $\mathscr{U}$ that commute with the action of K′, or equivalently, with the action of the diagonal subgroup $(g, \ldots, g)$ of $\underbrace{G \times \ldots \times G}_{r}$. Set

$$R_{\alpha\beta}^{(p_i)} = \sum_{k=1}^{p_i} Z_{k\alpha} \frac{\partial}{\partial Z_{k\beta}} \qquad 1 \leqslant \alpha, \beta \leqslant N \tag{3.3}$$

and let $R^{(p_i)}$ denote the matrix $(R_{\alpha\beta}^{(p_i)})$.

Let us write the matrix $[L] = (L_{ij})$, $1 \leqslant i, j \leqslant n$, in block form as

$$[L] = \begin{bmatrix} [L]_{11} & \cdots & [L]_{1r} \\ \vdots & & \vdots \\ [L]_{r1} & \cdots & [L]_{rr} \end{bmatrix} \tag{3.4}$$

where each $[L]_{uv}$ block is a $p_u \times p_v$ matrix $1 \leqslant u, v \leqslant r$. We now have the main theorem of this paper.

Theorem 3.1. In the universal enveloping algebra $\mathscr{U}$ the elements of the form

$$\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3} \ldots [L]_{u_qu_1}) \qquad 1 \leqslant u_j \leqslant r,\ 1 \leqslant j \leqslant q \tag{3.5}$$

where q is a positive integer and Tr denotes the trace of a matrix, generate a subalgebra of differential operators that commute with the action of the subgroup $\mathrm{K}' = \mathrm{GL}(p_1, \mathbb{C}) \times \ldots \times \mathrm{GL}(p_r\mathbb{C})$ on $\mathscr{F}(\mathbb{C}^{n\times N})$.

Proof. Let $\mathfrak{G}'$ denote the Lie algebra spanned by the basis elements L_{ij}, $1 \leqslant i, j \leqslant n$. Then each element X in $\mathfrak{G}'$ can be written uniquely as

$$X = \sum_{i,j=1}^{n} x_{ij} L_{ij}$$

and so X can be identified with the matrix $x=(x_{ij})$. Similarly we can consider elements of the dual $(\mathfrak{G}')^*$ of $\mathfrak{G}'$ as $n\times n$ matrices. Let S denote the symmetric algebra of all polynomial functions on $(\mathfrak{G}')^*$. Then $\mathfrak{G}'$ can be considered as the set of all polynomials of first degree (linear forms) on $(\mathfrak{G}')^*$ via the pairing $x(\xi)=\mathrm{Tr}(x\xi)$, for all (x,ξ) in $\mathfrak{G}'\times(\mathfrak{G}')^*$. Since the representation $x\to gxg^{-1}$, $g\in \mathrm{G}'$, is contragredient to the representation $\xi\to g^{-1}\xi g$ we can define the coadjoint representation T of G′ in S by the equation

$$[T(g)p](\xi)=p(g^{-1}\xi g) \qquad g'\in \mathrm{G}, p\in \mathrm{S}. \tag{3.6}$$

An element $p\in\mathrm{S}$ is said to be K′ invariant if $T(k)p=p$ for all $k\in\mathrm{K}'$.

We have now the canonical isomorphism Φ of S onto $\mathscr{U}$ (cf Dixmier 1974, ch 3) defined as follows.

Let p be an element of S; then p can be expressed uniquely as

$$p(\xi)=\sum_{s\leq d} a_{i_1,j_1\ldots i_s,j_s}\xi_{i_1,j_1}\ldots\xi_{i_s,j_s}$$

where the coefficients $a_{i_1,j_1\ldots i_s,j_s}$ are symmetric functions, i.e. $a_{i_{\sigma(1)}j_{\sigma(1)}\ldots i_{\sigma(s)}j_{\sigma(s)}}=a_{i_1,j_1\ldots i_s,j_s}$ for all permutations $\sigma\in\mathfrak{S}_S$ and for all integers s less than or equal to a fixed integer d. Now $\Phi: S\to\mathscr{U}$ is defined by

$$\Phi(p)=\sum_{s\leq d} a_{i_1,j_1\ldots i_s,j_s}L_{i_1j_1}\ldots L_{i_sj_s}.$$

An element $u\in\mathscr{U}$ is said to be Casimir K′ invariant if it commutes with the action of K′ on $\mathscr{F}$, or equivalently, if $[u, L_{i_pj_p}]=0$ for all $L_{i_pj_p}$. Since the $L_{i_pj_p}$ generate the Lie algebra of K′ it is also equivalent to say that u is K′ invariant if and only if u belongs to the centre of the universal enveloping algebra of K′. It is well known (see Dixmier 1974, ch 2) that the canonical isomorphism carries the K′-invariant polynomials onto the Casimir K′-invariant differential operators. Thus to show that a differential operator of the form (3.5) is K′ invariant it sufficies to show that its inverse image under the canonical map Φ is a K′-invariant polynomial function. For this we partition a matrix $\xi\in\mathbb{C}^{n\times n}$ in the same way as the matrix $[L]$ of (3.4), namely as

$$\xi=\begin{bmatrix}[\xi]_{11} & \cdots & [\xi]_{1r}\\ \vdots & & \vdots\\ [\xi]_{r1} & \cdots & [\xi]_{rr}\end{bmatrix} \tag{3.7}$$

where each $[\xi]_{uv}$ is a $p_u\times p_v$ matrix, $1\leq u$, $v\leq r$. Now let

$$k=\begin{bmatrix}k_1 & & \\ & \ddots & \\ & & k_r\end{bmatrix}$$

be an element of K′, $k_i\in\mathrm{GL}(p_i,\mathbb{C})$ $1\leq i\leq r$; then the (u,v) block matrix of $k^{-1}\xi k$ is $k_u^{-1}[\xi]_{uv}k_v$. If f is the inverse image under the the canonical isomorphism Φ then f is given by

$$f(\xi)=\mathrm{Tr}([\xi]_{u_1u_2}[\xi]_{u_2u_3}\ldots[\xi]_{u_qu_1}).$$

Now

$$\begin{aligned}[T](k)f](\xi)&=f(k^{-1}\xi k)\\ &=\mathrm{Tr}((k_{u_1}^{-1}[\xi]_{u_1u_2}k_{u_2})(k_{u_2}^{-1}[\xi]_{u_2u_3}k_{u_3})\ldots(k_{u_q}^{-1}[\xi]_{u_qu_1}k_{u_1}^{-1}))\\ &=\mathrm{Tr}(k_{u_1}^{-1}[\xi]_{u_1u_2}[\xi]_{u_2u_3}\ldots[\xi]_{u_qu_1}k_{u_1})\\ &=\mathrm{Tr}([\xi]_{u_1u_2}[\xi]_{u_2u_3}\ldots[\xi]_{u_qu_1})\\ &=f(\xi).\end{aligned}$$

This achieves the proof of the theorem.

Remark 3.2. The differential operators in (3.5) generalise the cycles defined by Želobenko (1970, ch 9, § 63) where a cycle of length q is defined as $L_{i_1 i_2} \dots L_{i_q i_1}$, $1 \leqslant i_j \leqslant n$, so that it corresponds to the particular case in our theorem when all the p_j are equal to 1.

Proposition 3.3. The differential operators of the form

$$\operatorname{Tr}((R^{(p_1)})^{d_1} \dots (R^{(p_r)})^{d_r}) \tag{3.8}$$

where the matrices $R^{(p_i)}$, $1 \leqslant i \leqslant r$, are defined by (3.3) and where each d_j, $1 \leqslant j \leqslant r$, is an integer $\geqslant 0$, generate the same algebra of K′-invariant differential operators as the differential operators defined by (3.5) in theorem 3.1.

Proof. A general proof of this proposition would involve induction on the degree of the differential operators and tedious computations. We elect instead to give a proof for a specific case and illustrate through this case the main steps involved in the general proof. For this we consider the Fock space $\mathscr{F}(\mathbb{C}^{n \times N})$ where $n = p_1 + p_2 + p_3$ so that a matrix Z is partitioned in block matrix of the form

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

where Z_i is a $p_i \times N$ matrix, $1 \leqslant i \leqslant 3$. Thus we only have three matrices, $R^{(p_1)}$, $R^{(p_2)}$ and $R^{(p_3)}$, and the corresponding matrix $[L]$ of (3.4) is of the form

$$[L] = \begin{bmatrix} [L]_{11} & [L]_{12} & [L]_{13} \\ [L]_{21} & [L]_{22} & [L]_{23} \\ [L]_{31} & [L]_{32} & [L]_{33} \end{bmatrix}.$$

Now let us consider the differential operator $\operatorname{Tr}(R^{(p_1)} R^{(p_2)} R^{(p_3)}]$ and show that it can be expressed in terms of the differential operators in (3.5). To avoid cumbersome sums in the traces of matrices we adopt the Einstein convention of summing over repeated indices. It follows that

$$\operatorname{Tr}(R^{(p_1)} R^{(p_2)} R^{(p_3)}) = Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} Z_{\beta j} \frac{\partial}{\partial Z_{\beta k}} Z_{\gamma k} \frac{\partial}{\partial Z_{\gamma i}}$$

where $1 \leqslant i, j, k \leqslant N$, $1 \leqslant \alpha \leqslant p_1$, $1 \leqslant \beta \leqslant p_2$, and $1 \leqslant \gamma \leqslant p_3$. Now we observe that

$$\frac{\partial}{\partial Z_{\gamma i}} (Z_{\gamma k}) = p_3 \delta_{ik} + Z_{\gamma k} \frac{\partial}{\partial Z_{\gamma i}}$$

where δ_{ik} is a Kronecker delta. Therefore,

$$Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} Z_{\beta j} \frac{\partial}{\partial Z_{\beta k}} Z_{\gamma k} \frac{\partial}{\partial Z_{\gamma i}} = Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} Z_{\beta j} \frac{\partial}{\partial Z_{\beta k}} \left(-p_3 \delta_{ik} + \frac{\partial}{\partial Z_{\gamma i}} Z_{\gamma k} \right)$$

$$= -p_3 Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} Z_{\beta j} \frac{\partial}{\partial Z_{\beta i}} + Z_{\alpha i} \frac{\partial}{\partial Z_{\gamma i}} \frac{\partial}{\partial Z_{\alpha j}} Z_{\beta j} Z_{\gamma k} \frac{\partial}{\partial Z_{\beta k}}.$$

We note that at this point in the general proof we would use induction on the first term of the second member of the last equation since it is of lower degree. Now the equations

$$p_1\delta_{ij} + Z_{\alpha i}\frac{\partial}{\partial Z_{\alpha j}} = \frac{\partial}{\partial Z_{\alpha j}}(Z_{\alpha i}) \qquad p_2\delta_{jk} + Z_{\beta j}\frac{\partial}{\partial Z_{\beta k}} = \frac{\partial}{\partial Z_{\beta k}}(z_{\beta j})$$

imply that the above equation is equal to

$$-p_3\left(-p_1\delta_{ij} + \frac{\partial}{\partial Z_{\alpha j}}(Z_{\alpha i})\right)Z_{\beta j}\frac{\partial}{\partial Z_{\beta i}} + L_{\alpha\gamma}Z_{\gamma k}\left(-p_2\delta_{jk} + \frac{\partial}{\partial Z_{\beta k}}(Z_{\beta j})\right)\frac{\partial}{\partial Z_{\alpha j}}$$
$$= p_3 p_1 L_{\beta\beta} - p_3 L_{\beta\alpha}L_{\alpha\beta} - p_2 L_{\alpha\gamma}L_{\gamma\alpha} + L_{\alpha\gamma}L_{\gamma\beta}L_{\beta\alpha}$$
$$= p_3 p_1 \operatorname{Tr}([L]_{22}) - p_3 \operatorname{Tr}([L]_{21}[L]_{12}) - p_2 \operatorname{Tr}([L]_{13}[L]_{31})$$
$$+ \operatorname{Tr}([L]_{13}[L]_{32}[L]_{21}).$$

Conversely, it is easy to verify that

$$\operatorname{Tr}([L]_{22}) = \operatorname{Tr}(R^{(p_2)})$$
$$\operatorname{Tr}([L]_{21}[L]_{12}) = \operatorname{Tr}(R^{(p_2)}R^{(p_1)}) + p_1 \operatorname{Tr}(R^{(p_2)})$$
$$\operatorname{Tr}([L]_{13}[L]_{31}) = \operatorname{Tr}(R^{(p_1)}R^{(p_3)}) + p_3 \operatorname{Tr}(R^{(p_1)})$$

so that

$$\operatorname{Tr}([L]_{13}[L]_{32}[L]_{21}) = \operatorname{Tr}(R^{(p_1)}R^{(p_2)}R^{(p_3)}) - p_3 p_1 \operatorname{Tr}(R^{(p_2)}) + p_3 \operatorname{Tr}(R^{(p_2)}R^{(p_1)})$$
$$+ p_3 p_1 \operatorname{Tr}(R^{(p_2)}) + p_2 \operatorname{Tr}(R^{(p_1)}R^{(p_3)}) + p_2 p_3 \operatorname{Tr}(R^{(p_1)}).$$

Thus, we have just shown that $\operatorname{Tr}(R^{(p_1)}R^{(p_2)}R^{(p_3)})$ can be expressed as a sum of $\operatorname{Tr}([L]_{13}[L]_{32}[L]_{21})$ and terms of lower degrees, and vice versa. Clearly, a general proof can proceed in a similar fashion by showing that $\operatorname{Tr}((R^{(p_1)})^{d_1}\ldots(R^{(p_r)})^{d_r})$ can be expressed as the sum of a differential operator of the same degree of the form $\operatorname{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ and terms of lower degrees which by induction satisfy the conclusion of the proposition. Note that in the first step of the induction for the terms of degree one we obviously have $\operatorname{Tr}(R^{(p_i)}) = \operatorname{Tr}([L]_{ii})$, $1 \leqslant i \leqslant r$. Thus proposition 3.3 is proved.

Remark 3.4. It is not difficult to show directly that the differential operators of the form $\operatorname{Tr}((R^{(p_1)})^{d_1}\ldots(R^{(p_2)})^{d_r})$ commute with the right action of the diagonal subgroup $(g, g, \ldots, g)$ of $\underbrace{G\times\ldots\times G}_{r}$, so that theorem 3.1 and proposition 3.3 illustrate indeed a dual pair action on $\mathscr{F}(\mathbb{C}^{n\times N})$.

To be useful for our programme of multiplicity breaking the commuting operators must be Hermitian. We will show this fact by making use of the differentiation inner product given by (2.1); this will also illustrate the advantage of the differentiation inner product over the integration inner product (2.2).

Proposition 3.5. The differential operators of the form $\operatorname{Tr}((R^{(p_1)})^{d_1}\ldots(R^{(p_r)})^{d_r})$ are Hermitian.

Proof. Recall that, using the Einstein convention, an operator $R_{lj}^{(p_i)}$, $1 \leq l, j \leq N$, is defined by

$$R_{lj}^{(p_i)} = Z_{kl} \frac{\partial}{\partial Z_{kj}} \qquad 1 \leq k \leq p_i, 1 \leq i \leq r.$$

In the appendix we show that, using the differentiation inner product, $(R_{lj}^{(p_i)})^* = R_{jl}^{(p_i)}$ for all $l, j = 1, \ldots, N$. Recall that $R^{(p_i)}$ denotes the matrix $(R_{lj}^{(p_i)})$, then it follows immediately that

$$\begin{aligned}[\mathrm{Tr}((R^{(p_1)})^{d_1} \ldots (R^{(p_2)})^{d_r})]^* &= \mathrm{Tr}((R^{(p_1)})^{d_1} \ldots (R^{(p_r)})^{d_r})^* \\ &= \mathrm{Tr}((R^{(p_r)*})^{d_r} \ldots (R^{(p_1)*})^{d_1}) = \mathrm{Tr}((R^{(p_r)})^{d_r} \ldots (R^{(p_1)})^{d_1}).\end{aligned} \tag{3.9}$$

From (3.3) we see that two matrix entries $R_{\alpha\beta}^{(p_i)}$ and $R_{\gamma\delta}^{(p_j)}$ with $i \neq j$ commute with each other since they involve different sets of variables. From this it follows immediately that

$$\mathrm{Tr}((R^{(p_r)})^{d_r} \ldots (R^{(p_1)})^{d_1}) = \mathrm{Tr}((R^{(p_1)})^{d_1} \ldots (R^{(p_r)})^{d_r})$$

and hence (3.9) implies that $\mathrm{Tr}((R^{(p_1)})^{d_1} \ldots (R^{(p_r)})^{d_r})$ is Hermitian.

Corollary 3.6. The differential operators of the form $\mathrm{Tr}([L]_{u_1 u_2} \ldots [L]_{u_q u_1})$ as given by (3.5) are Hermitian.

Proof. The proof is an immediate consequence of propositions 3.3 and 3.5.

For computational purposes the differential operators $\mathrm{Tr}([L]_{u_1 u_2} \ldots [L]_{u_q u_1})$ are more convenient than the differential operators $\mathrm{Tr}((R^{(p_1)})^{d_1} \ldots (R^{(p_r)})^{d_r})$. This fact will be illustrated in the next section where we will show how to resolve the multiplicity in two examples using the procedure just described.

4. Examples

4.1. Multiplicity breaking of the irreducible representation of GL(3, ℂ) with signature (3, 2, 1) in the tensor product $V^{(2,1,0)} \otimes V^{(2,1,0)}$

This is the well known tensor product of two eight-dimensional representations of U(3) in which the eight-dimensional representation of U(3) occurs twice. We will show that our procedure will also allow us to rederive this multiplicity. According to our programme we consider the Fock space $\mathscr{F}(\mathbb{C}^{4\times 3})$ which contains the GL(3, ℂ) module $\mathscr{P}^{(2,1,2,1)}(\mathbb{C}^{4\times 3})$. This module contains in turn the module $H^{(2,1,2,1)}(\mathbb{C}^{4\times 3})$ which by theorem 2.3 is isomorphic to the tensor product $V^{(2,1,0)} \otimes V^{(2,1,0)}$. The submodule $H^{(2,1,2,1)}$ consists of polynomial functions in $\mathscr{P}^{(2,1,2,1)}$ which are simultaneously annihilated by the lowering operators

$$L_{12} = \sum_{l=1}^{3} Z_{1l} \frac{\partial}{\partial Z_{2l}} \qquad L_{34} = \sum_{l=1}^{3} Z_{3l} \frac{\partial}{\partial Z_{4l}}.$$

According to theorem 2.1(*b*) the number of times that $V^{(3,2,1)}$ occurs in $\mathscr{P}^{(2,1,2,1)}$ is equal to the dimension of the weight space (2, 1, 2, 1) in $V^{(3,2,1,0)}$. To find this dimension we consider the Gelfand–Žetlin tableau

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & i_1 & & i_2 & & i_3 & \\ & & j_1 & & j_2 & & \\ & & & k_1 & & & \\ & & & & l & & \end{pmatrix}.$$

Then according to theorem 3, § 67, of Zelobenko (1970) a basis element labelled by the tableau above has weight (2, 1, 2, 1) if and only if

$$k=2 \qquad j_1+j_2=3 \qquad i_1+i_2+i_3=5$$

together with the betweeness conditions of a Gelfand-Žetlin tableau. This leads to the four possible tableaux

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 2 & & 0 & \\ & & 3 & & 0 & & \\ & & & 2 & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 2 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix}$$

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 1 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 2 & & 2 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix}.$$

Hence $V^{(3,2,1)}$ occurs in $\mathscr{P}^{(2,1,2,1)}$ with multiplicity 4. Set $f_0(Z)=\Delta_1^1(Z)\Delta_{12}^{12}(Z)\Delta_{123}^{123}(Z)$ where the Δ are principal minors of $Z\in\mathbb{C}^{4\times 3}$. There exist four linearly independent intertwining operators, for example, $L_{21}L_{32}^2L_{43}$, $L_{31}L_{32}L_{43}$, $L_{31}L_{42}$ and $L_{32}L_{41}$, that send the GL(3, $\mathbb{C}$) module $V^{(3,2,1)}$ into the GL(3, $\mathbb{C}$) module $\mathscr{P}^{(2,1,2,1)}$. Here L_{ij} is given by

$$L_{ij}=\sum_{l=1}^{3} Z_{il}\frac{\partial}{\partial Z_{jl}} \qquad 1\leqslant i,j\leqslant 4.$$

According to the scheme described in the previous section the matrix

$$[L]=\begin{bmatrix} [L]_{11} & [L]_{12} \\ [L]_{21} & [L]_{22} \end{bmatrix}$$

where

$$[L]_{11}=\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \qquad [L]_{12}=\begin{pmatrix} L_{13} & L_{14} \\ L_{23} & L_{24} \end{pmatrix}$$

$$[L]_{21}=\begin{pmatrix} L_{31} & L_{32} \\ L_{41} & L_{42} \end{pmatrix} \qquad [L]_{22}=\begin{pmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{pmatrix}.$$

We use the Casimir operator $C\equiv \mathrm{Tr}([L]_{12}[L]_{21}[L]_{12}[L]_{21})=\mathrm{Tr}(([L]_{12}[L]_{21})^2)=L_{is}L_{sk}L_{kr}L_{ri}$, $1\leqslant i,\ k\leqslant 2,\ 3\leqslant r,\ s\leqslant 4$, using the Einstein convention. The space $W_{\max}^{(3,2,1)(2,1,2,1)}$ is spanned by

$$f_1=L_{21}L_{32}^2L_{43}f_0 \qquad f_2=L_{31}L_{32}L_{43}f_0$$

$$f_3=L_{31}L_{42}f_0 \qquad \text{and} \qquad f_4=L_{32}L_{41}f_0.$$

The operators L_{12} and L_{34} then project $W_{\max}^{(3,2,1)(2,1,2,1)}$ onto $\ker_{\max}^{(312)(2,1,2,1)}$. The application of the operators L_{12} and L_{34} to a general vector in $W_{\max}^{(3,2,1)(2,1,2,1)}$ of the form $\Sigma_{i=1}^4 \alpha_i f_i$, $\alpha_i\in\mathbb{C}$, leads to a system of linear equations which in turn implies that $\ker_{\max}^{(3,2,1)(2,1,2,1)}$ has dimension 2; hence, the multiplicity of (3, 2, 1) in $V^{(2,1,0)}\otimes V^{(2,1,0)}$ is indeed 2. The Casimir operator C acting on $\ker_{\max}^{(312)(2,1,2,1)}$ has two distinct eigenvalues $\lambda_1=24$ and $\lambda_2=42$ with corresponding eigenvectors

$$h_1=f_1+2f_2-2f_3 \qquad h_2=f_1+2f_2+f_3-3f_4$$

which are obviously orthogonal since $\lambda_1 \neq \lambda_2$ and C is Hermitian. In conclusion, the two intertwining maps that send $V^{(3,2,1)}$ into two orthogonal (equivalent) submodules of $H^{(2,1,2,1)}$ are

$$P_1 = L_{21} L_{32}^2 L_{43} + 2 L_{31} L_{32} L_{43} - 2 L_{31} L_{42}$$

and

$$P_2 = L_{21} L_{32}^2 L_{43} + 2 L_{31} L_{32} L_{43} + L_{31} L_{42} - 3 L_{32} L_{41}$$

which are obtained from the forms of h_1 and h_2 in terms of the L_{ij}.

4.2. Multiplicity breaking of the irreducible representation of GL(3, ℂ) with signature (3, 2, 1) in the threefold tensor product $V^{(1,0,0)} \otimes V^{(1,1,0)} \otimes V^{(2,1,0)}$

For this example the Fock space is $\mathscr{F}(\mathbb{C}^{5\times 3})$ ($p_1 = 1$, $p_2 = p_3 = 2$ and $n = 5$) which contain the GL(3, ℂ) module $\mathscr{P}^{(1,1,1,2,1)}(\mathbb{C}^{5\times 3})$. The submodule $H^{(1,1,1,2,1)}(\mathbb{C}^{5\times 3})$ which is isomorphic to $V^{(1,0,0)} \otimes V^{(1,1,0)} \otimes V^{(2,1,0)}$ consists of polynomial functions in $\mathscr{P}^{(1,1,1,2,1)}$ that are simultaneously annihilated by the lowering operators

$$L_{23} = \sum_{l=1}^{3} Z_{2l} \frac{\partial}{\partial Z_{3l}} \quad \text{and} \quad L_{45} = \sum_{l=1}^{3} Z_{4l} \frac{\partial}{\partial Z_{5l}}.$$

To find the multiplicity of $V^{(3.2.1)}$ in $\mathscr{C}^{(1,1,1,2,1)}$ we consider the Gelfand–Žetlin tableau

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & i_1 & & i_2 & & i_3 & & 0 & \\ & & j_1 & & j_2 & & j_3 & & \\ & & & k_1 & & k_2 & & & \\ & & & & l & & & & \end{pmatrix}.$$

Then a basis element labelled by this tableau has weight $(1, 1, 1, 2, 1)$ if and only if $l = 1$, $k_1 + k_2 = 2$, $j_1 + j_2 + j_3 = 3$, and $i_1 + i_2 + i_3 = 5$, together with the betweeness conditions of a Gelfand–Žetlin tableau. This leads to the eight possible tableaux

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 1 & & 0 & \\ & & 1 & & 1 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 2 & & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 2 & & 0 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{pmatrix}$$

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 1 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 2 & & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 2 & & 0 & & & \\ & & & & 1 & & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 2 & & 0 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 2 & & 0 & & & \\ & & & & 0 & & & & \end{pmatrix}$$

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 1 & & 0 & \\ & & 2 & & 1 & & 0 & & \\ & & & 2 & & 0 & & & \\ & & & & 1 & & & & \end{pmatrix} \quad \begin{pmatrix} 3 & & 2 & & 1 & & 0 & & 0 \\ & 3 & & 2 & & 0 & & 0 & \\ & & 3 & & 0 & & 0 & & \\ & & & 2 & & 0 & & & \\ & & & & 1 & & & & \end{pmatrix}.$$

Hence $V^{(3,2,1)}$ occurs in $\mathcal{P}^{(1,1,1,2,1)}$ with multiplicity 8. Note that it is important to exhibit the eight tableaux above because they allow us to get eight linearly independent intertwining maps that send $V^{(3,2,1)}$ into $\mathcal{P}^{(1,1,1,2,1)}$, namely

$$L_{41}^2L_{52} \qquad L_{31}L_{42}L_{43}L_{51} \qquad L_{31}L_{44}L_{42}L_{53}$$
$$L_{31}L_{41}L_{43}L_{52} \qquad L_{21}L_{32}L_{42}L_{43}L_{51} \qquad L_{21}L_{32}L_{41}L_{42}L_{53}$$
$$L_{21}L_{32}L_{41}L_{53} \qquad \text{and} \qquad L_{21}L_{31}L_{42}^2L_{53}.$$

Set $f_0 = h_{\max}^{(321)}$ and let f_i, $1 \leqslant i \leqslant 8$, be the images of f_0 under the intertwining maps above. Then the vector space $W_{\max}^{(3,2,1)(1,1,1,2,1)}$ is spanned by the f_i. The equations

$$L_{23}\left(\sum_{i=1}^{8} \alpha_i f_i\right) = L_{45}\left(\sum_{i=1}^{8} \alpha_i f_i\right) = 0 \qquad \alpha \in \mathbb{C}$$

lead to a system of linear equations which in turn implies that $\ker_{\max}^{(3,2,1)(1,1,1,2,1)}$ has dimension 3; hence, the multiplicity of $(3,2,1)$ in $V^{(1,0,0)} \otimes V^{(1,1,0)} \otimes V^{(2,1,1)}$ is 3. The Casimir operator $C = L_{is}L_{sk}L_{kr}L_{ri}$; $2 \leqslant i, k \leqslant 3$, $4 \leqslant r, s \leqslant 5$, acting on $\ker_{\max}^{(3,2,1)(1,1,1,2,1)}$ has three distinct eigenvalues $\lambda_1 = 9$, $\lambda_2 = 29$ and $\lambda_3 = 35$. The corresponding mutually orthogonal eigenvectors are

$$h_1 = -6f_1 - 8f_2 - 10f_3 + 2f_4 + 8f_5 + 6f_6 - 2f_7 + 3f_8$$
$$h_2 = 2f_1 - f_3 + f_4 - 3f_6 - f_7 - f_8$$
$$h_3 = -6f_1 + 4f_2 - 2f_3 - 2f_4 - 4f_5 + 14f_6 + 2f_7 + 3f_8.$$

Therefore, the three intertwining operators that send $V^{(3,2,1)}$ into three (equivalent) submodules of $H^{(1,1,1,2,1)}$ are

$$P_1 = -6L_{41}^2L_{52} - 8L_{31}L_{42}L_{43}L_{51} - 10L_{31}L_{41}L_{42}L_{53} + 2L_{31}L_{41}L_{43}L_{52} + 8L_{21}L_{32}L_{42}L_{43}L_{51}$$
$$+ 6L_{21}L_{32}L_{41}L_{42}L_{53} - 2L_{21}L_{32}L_{41}L_{43}L_{53} + 3L_{21}L_{31}L_{42}^2L_{53}$$
$$P_2 = 2L_{41}^2L_{53} - L_{31}L_{41}L_{42}L_{53} + L_{31}L_{41}L_{43}L_{52} - 3L_{21}L_{32}L_{41}L_{42}L_{53}$$
$$- L_{21}L_{32}L_{41}L_{43}L_{53} - L_{21}L_{31}L_{42}^2L_{53}$$
$$P_3 = -6L_{41}^2L_{53} + 4L_{31}L_{42}L_{43}L_{51} - 2L_{21}L_{41}L_{42}L_{53} - 2L_{31}L_{41}L_{43}L_{52} - 4L_{21}L_{32}L_{42}L_{43}L_{51}$$
$$+ 14L_{21}L_{32}L_{42}L_{53} + 2L_{21}L_{32}L_{41}L_{43}L_{53} + 3L_{21}L_{31}L_{42}^2L_{53}.$$

5. Conclusion

We have shown how to decompose an r-rold tensor product of arbitrary irreducible representations of the U(N) groups, by finding generalised Casimir operators whose eigenvalues can be used to resolve the ambiguity occurring when equivalent representations appear more than once in the decomposition. The procedure given is computationally effective, in that maps are constructed which take an irreducible representation space into the tensor product space, resulting in an orthogonal direct sum decomposition of equivalent representations.

Underlying our procedure is the use of polynomial realisations of all the irreducible representations of the U(N) groups. Such polynomial realisations have the advantage of being basis independent; different bases, dictated by physical considerations, result in different sets of polynomials, and the transformation coefficients between the basis sets are easily calculated using the differentiation inner product (2.1). The maps that take a polynomial from an irreducible representation space to the tensor product space may result in polynomials that are quite long and complicated. Our goal is not to find closed form expressions for coefficients of physical interest (such as Clebsch-Gordan and Racah coefficients) but rather to give well defined procedures that can be adapted to the computer. We have shown (Klink and Ton-That 1988b) how to give procedures for calculating coefficients of physical interest for the simplest representations of $U(N)$, of the form $(M, 0, \ldots, 0)$. In the following paragraphs we summarise how those procedures are generalised to arbitrary irreducible representations of $U(N)$.

We assume that an r-fold tensor product of representations (M_1), $(M_2), \ldots, (M_r)$ of $U(N)$ is given; the goal is to give an orthogonal direct sum decomposition of the r-fold tensor product into irreducible representations of $U(N)$. This is equivalent to specifying the maps that take an irreducible representation space labelled by (m) to an orthogonal direct sum of copies of (m) in the tensor product space. Our procedure begins by arranging the labels (M_1), $(M_2), \ldots, (M_r)$ as an n-tuple of integers by discarding the zero entries in the (M_i). For example, the tensor product of the eight-dimensional representation of U(3) with itself, $(2, 1, 0)\otimes(2, 1, 0)$, discussed in § 4.1 goes to $(M) = (2, 1, 2, 1)$, with $n = 4$.

We next introduce the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$; the action of $U(N) \subset GL(N, \mathbb{C})$ on $f \in \mathcal{F}$ is given by right translation, namely $(R(g)(f)(Z) = f(Zg), g \in GL(N, \mathbb{C})$. Actually only a finite-dimensional subspace of $\mathcal{F}(\mathbb{C}^{n\times N})$ is needed, namely the subspace $P^{(M)}$ of elements of $\mathcal{F}$ satisfying the covariance condition $f(dZ) = d_1^{M_1} \ldots d_n^{M_n} f(Z)$ (for the definition of d see the paragraph preceding theorem 2.1). From Klink and Ton-That (1988a) we can compute the number of times the representation (m) of $U(N)$ will occur in $\mathcal{P}^{(M)}$; it is given by certain Gelfand-Žetlin tableaux, as discussed in theorem 2.1. The maps which send the polynomial representation space labelled by (m) into $\mathcal{P}^{(M)}$ are obtained from the dual group $GL(n, \mathbb{C})$, whose (left) action $(L(g')f)(Z) \equiv f(g'^{-1}Z)$, $g' \in GL(n, \mathbb{C})$, commutes with the previously defined right action. The maps are certain polynomials in the Lie algebra generated by $L_{GL(n,\mathbb{C})}$, with the number of linearly independent maps given by the multiplicity of (m) in $\mathcal{P}^{(M)}$. These results are spelled out in more detail in Klink and Ton-That (1988b).

Thirdly the r-fold tensor product space is shown (theorem 2.3) to be isomorphic to a subspace $H^{(M)}$ of $\mathcal{P}^{(M)}$, defined to be those elements of $\mathcal{P}^{(M)}$ which are annihilated by certain elements in the Lie algebra of $L_{GL(n,\mathbb{C})}$ (see equations (2.3) and (3.2)). In particular the representation space (m), after being mapped into $\mathcal{P}^{(M)}$, is projected into $H^{(M)}$. So at this point we have the right number of copies of (m) in the tensor product space, but not yet as an orthogonal direct sum.

The main result of this paper has been to construct an algebra of operators out of the Lie algebra of $L_{GL(n,\mathbb{C})}$ which commutes with $U(N)$ and leaves the space $H^{(M)}$ invariant. The operators defined in theorem 3.1 are Hermitian (proposition 3.5) and so their eigenvalues can be used to form an orthogonal direct sum of the copies of (m) in $H^{(M)}$. The algebra of operators generalises the notion of coupling schemes used to break the multiplicity in Klink and Ton-That (1988b).

The procedure outlined in the four steps above is not completely computationally effective. It has been shown (Procesi 1976) that al the generalised Casimir operators can be generated from a finite set; in a succeeding paper we will show how to choose this set. Also, we have been somewhat vague on how to construct the maps carrying (m) into $\mathcal{P}^{(M)}$. When writing our procedures as instructions for a computer, we will show that the Gelfand tableaux used to calculate the multiplicity can also be used to specify the linearly independent maps. Finally it remains to find closed-form expressions for the eigenvalues of the generalised Casimir operators, analogous to the expressions that Želobenko (1970) has for the eigenvalues of his Casimir operators.

Though we have restricted our attention in this paper to the unitary groups, our procedure can be used on any groups whose irreducible representations can be realised as polynomials. Recently we have found that all the irreducible representations of the orthogonal and symplectic groups can be realised as polynomials; we intend to investigate the decomposition of tensor products for these groups in a future publication.

Appendix

In this appendix we will show that the adjoint R_{ij}^* of the operator R_{ij} is the operator R_{ji}. Let us recall that the differentiation inner product is defined on the Fock space $\mathscr{F}(\mathbb{C}^{n\times N})$ by the equation

$$(f, f') = f(D)\overline{f'(\bar{Z})}|_{Z=0}$$

for all f, f' in $\mathscr{F}$ and all $Z \in \mathbb{C}^{n\times N}$. Recall that, using the Einstein convention, an operator R_{ij}, $1 \leqslant i, j \leqslant N$, is defined by $R_{ij} = Z_{ki}\partial/\partial Z_{kj}$, where k ranges over an appropriate set of indices. Since the algebra $\mathscr{P}(\mathbb{C}^{n\times N})$ of all polynomial functions on $\mathbb{C}^{n\times N}$ is dense in $\mathscr{F}$ it follows that we will reach the conclusion $R_{ij}^* = R_{ji}$ by showing that

$$(f, R_{ij}f') = (R_{ji}f, f') \qquad \text{for all } f, f' \in \mathscr{P}(\mathbb{C}^{n\times N}).$$

If f and f' are linear forms (polynomials of degree one) this is a straightforward verification. Therefore, by induction, let us assume the formula true for all homogeneous polynomials of degree n, and consider polynomials f and f' of the form

$$f = f_1 f_2 \qquad f' = f_1' f_2'$$

where f_2 and f_2' are linear forms and f_1 and f_1' are homogeneous polynomials of degree n. Obviously

$$R_{ij}[f_1'f_2'](Z) = (R_{ij}f_1')(Z)f_2'(Z) + f_1'(Z)(R_{ij}f_2')(Z)$$

so that

$$\begin{aligned}(f_1f_2, R_{ij}[f_1'f_2')] &= f_1(D)f_2(D)\overline{\{(R_{ij}f_1')(\bar{Z})f_2'(\bar{Z}) + f_1'(\bar{Z})(R_{ij}f_2'(\bar{Z}))\}}|_{Z=0} \\ &= f_1(D)\{(f_2(D)\overline{R_{ij}f_1'})(\bar{Z})\overline{f_2'(\bar{Z})} + \overline{R_{ij}f_1'}(\bar{Z})f_2(D)\overline{f_2'(\bar{Z})} \\ &\quad + (f_2(D)\overline{f_1'(\bar{Z})})\overline{R_{ij}f_2'(\bar{Z})} + \overline{f_1'(\bar{Z})}f_2(D)\overline{R_{ij}f_2'(\bar{Z})}\}|_{Z=0}.\end{aligned}$$

By observing that $f_2(D)\overline{f_2'(\bar{Z})} = (f_2, f_2')$ and $f_2(D)\overline{R_{ij}f_2'(\bar{Z})} = (f_2, R_{ij}f_2')$ we obtain

$$\begin{aligned}(f_1f_2, R_{ij}[f_1'f_2']) &= f_1(D)\{(f_2(D)\overline{R_{ij}f_1'})(\bar{Z})\overline{f_2'(\bar{Z})} + (f_2(D)\overline{f_1'(\bar{Z})})\overline{R_{ij}f_2'(\bar{Z})}\}|_{Z=0} \\ &\quad + (f_1, R_{ij}f_1')(f_2, f_2') + (f_1, f_1')(f_2, R_{ij}f_2'). \end{aligned} \tag{A1}$$

A similar computation shows that

$$\begin{aligned}(R_{ij}[f_1f_2], f_1'f_2') &= (R_{ji}f_1)(D)\{f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0} \\ &\quad + f_1(D)\{(R_{ji}f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0} \\ &\quad + (R_{ji}f_1, f_1')(f_2, f_2') + (f_1, f_1')(R_{ji}f_2, f_2'). \end{aligned} \tag{A2}$$

By induction we have $(f_1, R_{ji}f_1') = (R_{ij}f_1, f_1')$ and $(f_2, R_{ij}f_2') = (R_{ji}f_2, f_2')$, so that to obtain the desired result it suffices to show in (A1) and (A2) that

$$f_1(D)\{(f_2(D)\overline{R_{ij}f_1'(\bar{Z})})\overline{f_2'(\bar{Z})} + (f_2(D)\overline{f_1'(\bar{Z})})\overline{R_{ij}f_2'(\bar{Z})}\}|_{Z=0}$$
$$= R_{ji}f_1'(D)\{(f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0} + f_1(D)\{(R_{ji}f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0}. \tag{A3}$$

Now by induction,

$$(R_{ji}f_1)(D)\{(f_2(D)\overline{f_1'(\bar{Z})}\overline{f_2'(\bar{Z})}\}|_{Z=0}$$
$$= f_1(D)\{R_{ij}((f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})})\}|_{Z=0}$$
$$= f_1(D)\{R_{ij}(f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})} + (f_2(D)\overline{f_1'(\bar{Z})})R_{ij}\overline{f_2'(\bar{Z})}\}|_{Z=0}.$$

Now $f_2(D)\overline{R_{ij}f_1'(\bar{Z})} = (R_{ji}f_2)(D)\overline{f_1'(\bar{Z})} + R_{ij}(f_2(D)\overline{f_1'(\bar{Z})})$ so that

$$f_1(D)\{(f_2(D)\overline{R_{ij}f_1')(\bar{Z})}\overline{f_2'(\bar{Z})}\}|_{Z=0}$$
$$= f_1(D)\{((R_{ji}f_2)(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0} + f_1(D)\{R_{ij}(f_2(D)\overline{f_1'(\bar{Z})})\overline{f_2'(\bar{Z})}\}|_{Z=0}.$$

This establishes (A3), and hence the induction is completed.

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